Solutions

1.2: Propositional Logic

You may have noticed that, in the truth table approach, every time you add a simple statement the number of entries doubles. That is to say, if a compound statement is made up of n simple statements, then the truth table has 2^n entries. (Does this remind you of computers somehow?) In this section we will develop the propositional calculus which will allow us to make logical deductions without resorting to the truth table. Not only does propositional calculus make it possible to analyze complex logical statements when truth tables are impractical, but the derivation rules are also commonly used in mathematical discourse and will provide us with a simple example of mathematical proof.

Question 1. How are the following questions related?

If you pass all the exams, will you pass the course?

Is it possible to pass all the exams and fail the course?

Same question just with truth values reversed.

Question 2. Consider the following statement.

If you have a ticket, then, as long as you are wearing a shirt, you may enter the theater, unless you aren't wearing shoes.

Write a simpler statement that expresses the same policy. Explain how you know that your statement is equivalent.

If you have a ticket and are wearing a shirt and shoes, you may enter the theater.

Question 3. Suppose that a natural number n is *gaunt* if it satisfies the following condition.

If n is even, then 10 divides n, and, if n is odd, then 5 divides n.

List the first 6 gaunt numbers. Is there a simpler way to define "gauntness?"

5, 10, 15, 20, 25, 30

n is gaunt if 5 divides n.

Definition 1. A statement which is true in all cases is called a tautology.

Example 1.

(a) $p \vee \neg p$

(b) $(p \wedge q) \rightarrow p$

(c) $\neg(p \vee q) \Longleftrightarrow \neg p \wedge \neg q$

(d) $\neg(p \wedge q) \Longleftrightarrow \neg p \vee \neg q$

Whenever an implication statement $A \rightarrow B$ is a tautology, the textbook writes $A \Rightarrow B$. Similarly, if an if and only if statement $A \leftrightarrow B$ is a tautology, then the textbook writes $A \iff B$. When a tautology is of the form $(C \wedge D) \Rightarrow E$, the textbook writes

$$\left.\begin{matrix} C \\ D \end{matrix}\right\} \Rightarrow E$$

so that the $\wedge$ connective is implicit.

Example 2. Use a truth table to prove the following.

$$\left.\begin{matrix} p \\ p \rightarrow q \end{matrix}\right\} \Rightarrow q$$

p	q	$p \rightarrow q$	$p \wedge (p \rightarrow q)$	$[p \wedge (p \rightarrow q)] \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

✓

Definition 2. A statement which is false in all cases is called a contradiction.

Example 3. $p \wedge \neg p$ is a contradiction. Can you come up with any others?

The negation of any tautology.

If a statement is "contingent" on the T/F inputs; i.e. not a tautology or a contradiction, then it is called a contingency.

Derivation Rules. Suppose that the following statements are true.

> Our professor does not own a spaceship.
> If our professor is from Mars, then our professor owns a spaceship.

Then we can deduce that our professor is not from Mars. This is true because of the contrapositive tautology $p \to q \iff \neg q \to \neg p$.

Equivalence	Name
$p \iff \neg\neg p$	double negation
$p \to q \iff \neg p \vee q$	implication
$\neg(p \vee q) \iff \neg p \wedge \neg q$ $\neg(p \wedge q) \iff \neg p \vee \neg q$	De Morgan's laws
$p \vee q \iff q \vee p$ $p \wedge q \iff q \wedge p$	commutativity
$p \wedge (q \wedge r) \iff (p \wedge q) \wedge r$ $p \vee (q \vee r) \iff (p \vee q) \vee r$	associativity

There are two ways to use an equivalence rule of the form $A \iff B$. First, given A, deduce B or, equivalently (pun intended), given B, deduce A. The second is a form of *substitution*; given a statement containing A, deduce the same statement, but with A replaced by B. ~~It is worth noting that $\iff$ has not replaced the symbol $\sim$ from the previous section.~~

Inference	Name
$\left.\begin{array}{l} p \\ q \end{array}\right\} \Rightarrow p \wedge q$	conjunction
$\left.\begin{array}{l} p \\ p \to q \end{array}\right\} \Rightarrow q$	*modus ponens* (basic inference)
$\left.\begin{array}{l} \neg q \\ p \to q \end{array}\right\} \Rightarrow \neg p$	contrapositive or *modus ponens*
$p \wedge q \Rightarrow p$	simplification
$p \Rightarrow p \vee q$	addition

Proof Sequences. We will use the derivation rules above to introduce proof sequences without resorting to truth tables.

Example 4. Write a proof sequence for the assertion $\left.\begin{array}{c} p \\ p \to q \\ q \to r \end{array}\right\} \Rightarrow r.$

Statements	Reasons
1. p	given
2. $p \to q$	given
3. $q \to r$	given
4. q	~~implication~~, 1, 2, basic inference
5. r	~~implication~~, 4, 3, basic inference

Example 5. Prove $\left.\begin{matrix} p \vee q \\ \neg p \end{matrix}\right\} \Rightarrow q.$

Statements	Reasons
1. $p \vee q$	given
2. $\neg p$	given
~~3. $\neg(\neg p) \vee q$~~	~~double negation, 1~~
~~3~~4. $\neg p \to q$	implication, 1
4. q	basic inference, 2, ~~3~~

technically should be

1. $p \vee q$	given
2. $\neg p$	given
3. $\neg(\neg p) \vee q$	double neg, 1
4. $\neg p \to q$	implication, 3
5. q	basic inference 2, 4

Example 6. Prove $p \to q \Rightarrow \neg q \to \neg p$.

Sometimes it is helpful to work in both directions.

Forward

Statements	Reasons
1. $p \to q$	given
2. $\neg p \vee q$	implication, 1 (only option)

Backward *

Statements	Reasons
7. $\neg p \vee q$	Commutativity, ~~below~~ 8
8. $q \vee \neg p$	implication, ~~below~~ 9
9. $\neg(\neg q) \vee \neg p$	implication, ~~End result~~ 10
End result: (10) $\neg q \to \neg p$	What we want

Notice that 2 and 7 are the same. So we can combine them to make a direct proof sequence.

Homework. (Due Sept 10, 2018) Section 1.2: 2, 4, 8, 10, 16

Practice Problems. Section 1.2: 1-9 (odd), 15-18, **23**, **25-28**

* The backwards direction only works with equivalences, no inferences can be used. That would be like assuming what you have to prove.